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Irreversibility and Limiting Possibilities of Macrocontrolled Systems: II. Microeconomics*

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Abstract. The methods of finite time thermodynamics are used to solve the maximal profit problem in irreversible microeconomic systems, similar to the problems of generalized exergy, which were solved in the first part of the paper. The condition of equilibrium in open microeconomic systems is obtained.

1. Introduction

The existence of deep analogy between thermodynamic and economic systems was pointed out by Samuelson in his Nobel lecture [8]. For equilibrium processes this analogy has been studied in detail in [2, 3, 7]. Martinas successfully traces the analogy between basic notions of irreversible thermodynamics and microeconomics [4, 5, 6]. Very few results were obtained using the methods of finite-time thermodynamics [1].

In microeconomics the stocks of resource and capital (basic resource) of the economic agent play the role of intensive variables in thermodynamics (volume, internal energy, enthalpy, entropy, etc). Intensive variables in thermodynamics are temperatures, pressures, chemical potentials, etc. In microeconomics such variables are the prices (estimates) of resources. If two or more systems with the same parameters merge in thermodynamics or economics then their intensive variables do not change and their extensive variables add up. What is especially important is irreversibility that is characteristic for both thermodynamic and economic processes. In thermodynamics there is a measure of irreversibility — entropy increase in a closed system (energy dissipation). Entropy increase also determines decrease of system exergy. According to the second law of thermodynamics the entropy of the close system can only increase and its exergy decrease. $\boxed{\text{extensive?}}$

It is also possible to introduce a quantitative measure of irreversibility in economics, which describes the decreasing ability to derive capital from economic

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system. We shall call this measure dissipation (capital dissipation), in analogy with thermodynamics. Capital dissipation is related to the rates of resource exchange processes.

The capital (basic resource) plays a special role among other types of resources in economics, because it can be freely exchanged for any other resource. Basic resource is similar to work. Just as it is done in thermodynamics, it is possible to introduce economic index which characterizes the maximal-possible amount of basic resource E^0 that can be derived from N units of resource during resource-exchange and the maximal amount of capital E that can be derived from a system, which includes no more than one market of perfect competition. We shall call these values the *profitability* of the resource and the profitability of the system. Similarly to how dissipation in thermodynamics reduces exergy of the heat, capital dissipation reduces the profitability of the system.

Table 1 (an extended version of the Table in [7]) shows analogies between the elements of thermodynamic and microeconomic systems and their characteristics.

The following notations are used here: T_- and T are the temperatures of the market and the working body which contacts it; p_- is the estimate of the resource on the market with perfect competition; c is the price of the resources set by the intermediary; N the amount of resource, U is the internal energy of the system; q and g are the fluxes of heat and resource. If the system includes two market with perfect competition, p_+ and p_-, and $p_+ > p_-$, then it is clear that the value of resource is

$$E^0 = N(p_+ - p_-)\,. \tag{1}$$

If there are no restrictions on the rate e of profit generation then E^0 has the same value. If such restrictions are present then the limiting value of the derived capital from the flow of resource E_e will be lower than E^0. The same is also true for the maximal-possible amount of capital extractable from the system of economic agents (EA) in a limited time. We denote this value E_τ.

The second part of this paper is devoted to the calculation of profitability and to the discussion of the factor of irreversibility in microeconomics.

2. Profitability: Limiting Possibility to Extract Profit in a Closed Microeconomic System

Consider an economic system, which consists of k economic agents (EA). Each EA has the stock of resource (commodity) N_i $(i = 1, 2, \ldots, k)$ and the stock of capital (basic resource) M_i. The resource estimate p_i depends on N_i and M_i. It is defined as the minimal price for which EA is ready is sell the resource and the maximal price, for which it is ready to buy it. A market can be one of the subsystems, for which the resource estimate p_- is constant and does not depend on its stock.

We assume that the system considered is closed, that is, the resource exchange between the system and the environment cannot occur but the capital exchange with the environment is possible. During a contact between i-th and j-th subsystems the fluxes of resource n_{ij} and capital n^0_{ij} occur. The resource flux is directed

Thermodynamic system			Microeconomic system	
N	Name	Denotation	Name	Denotation
1.	Reservoir (reversible heat exchange)	T_-	Market with perfect competition	p_-
2.	Reservoir (irreversible heat exchange)	T_- $q = \alpha(T, T_-)(T - T_-)$	Monopolistically competitive market	p_- $g = \alpha(p, p_-)(p - p_-)$
3.	System energy	U	Resource stock	N
4.	System with finite capacity, temperature	$T(U)$	Economic agent, resource estimate	$p(N)$
5.	Heat engine	$T(t)$	Trading firm, price	$c(t)$
6.	Mechanical energy	A	Basic resource	M
7.	System exergy	E	System profitability	E

Table 1: The analogies between thermodynamic and microeconomic systems.

from the subsystem with the lower estimate to the subsystem with the higher estimate. The flux of capital goes in the opposite direction to the flux of resource.

A system contains an economic intermediary (trading firm). The goal of this intermediary is to organize resource exchange in such a way that it extracts from the system some amount of capital M. We will also assume that the direct exchange of resources between EAs is not possible, the intermediary attempts to maximize M by setting the prices for buying and selling; and the fluxes of sold and bought resource depend on the price c_i offered by the intermediary to the i-th subsystem and on the resource estimate p_i by the i-th subsystem. Thus,

$$n_i = n_i(p_i, c_i)\,, \quad n_i = 0 \quad \text{for} \quad p_i = c_i\,, \quad \text{sign}\,(n_i) = \text{sign}\,(c_i - p_i)\,. \tag{2}$$

We assume that the flux is positive if it is directed to the intermediary. It is clear that the capital flux

$$n_i^0(p_i, c_i) = -c_i n_i(p_i, c_i) \,. \tag{3}$$

The stocks of resource and capital in the i-th subsystem change according to the equations

$$\begin{aligned} \dot{N}_i &= -n_i(p_i, c_i)\,, \quad N_i(0) = N_{i0}\,, \\ \dot{M}_i &= c_i n_i(p_i, c_i)\,, \quad M_i(0) = M_{i0}\,. \end{aligned} \tag{4}$$

Assume that for the fixed M_i the estimates $p_i(N_i, M_i)$ are functions which decrease monotonically when N_i increases. As a rule, for fixed N_i these functions are increasing functions of M_i. But it is also possible that the estimate does not depend on the amount of capital in the subsystem.

We will later find what are the volumes of capital, which intermediary can extract from the system if it operates in infinite time, and in finite time. These results will be obtained for systems, which include market and those which do not include it.

2.1. The duration of the process is not limited

2.1.1. System with a market

If we denote the resource's market price as p_-, then when $t \to \infty$ the resource estimate in any of the subsystems tends to p_-. From the conditions of equilibrium for $t \to \infty$, we get

$$p_i(\overline{N}_i, \overline{M}_i) = p_-\,, \qquad i = 1, \ldots, k\,. \tag{5}$$

Here $\overline{N}_i$, $\overline{M}_i$ denote the equilibrium stocks of resource and capital.

If there are no restrictions on the duration of operations then the intermediary buys resource for the price as much as close to p_i, and

$$\frac{dM_i}{dN_i} = -p_i(N_i, M_i)\,, \qquad M_i(N_{i0}) = M_{i0}\,. \tag{6}$$

The profitability of economic system — the limiting amount of extracted capital — here is

$$E_\infty = \sum_{i=1}^{k}(M_{i0} - \overline{M_i}) = \sum_{i=1}^{k} \int_{N_{i0}}^{\overline{N}_i} p_i(N_i, M_i(N_i))\, dN_i\,. \tag{7}$$

The conditions (57), (58) determine $2k$ unknowns $\overline{N_i}$ $\overline{M_i}$, and also E_∞.

EXAMPLE 1 Consider a system, which consists of two EAs and a market. The initial stocks of resources and capital for EAs N_{i0}, M_{i0}, $i = 1, 2$, are given, and the market price (the estimate of the resource on the market) p_- is given.

Let the resource estimates for the EAs have the form

$$p_i \;=\; \alpha_i \frac{M_i}{N_i}\,, \qquad i = 1,2\,. \tag{8}$$

The system (57) can be rewritten as

$$\frac{dM_i}{dN_i} \;=\; -\alpha_i \frac{M_i}{N_i}\,, \qquad M_i(N_{i0}) \;=\; M_{i0}\,,$$

which gives $M_i(N_i)$

$$M_i \;=\; \frac{M_{i0} \cdot N_{i0}^{\alpha_i}}{N_i^{\alpha_i}}\,, \qquad i = 1,2\,. \tag{9}$$

Let us find the equilibrium stocks of resource $\overline{N}_1$ and $\overline{N}_2$ from the conditions (56). After taking into account (59), (60), these conditions take the form

$$\alpha_i \frac{M_{i0} \cdot N_{i0}^{\alpha_i}}{N_i^{\alpha_i+1}} \;=\; p_-\,, \qquad i = 1,2\,.$$

We get

$$\overline{N}_i \;=\; \left(\frac{\alpha_i}{p_-} M_{i0} \cdot N_{i0}^{\alpha_i}\right)^{\frac{1}{\alpha_i+1}}, \qquad i = 1,2\,,$$

and the corresponding equilibrium stocks of capital are

$$\overline{M}_i \;=\; \frac{p_-}{\alpha_i} \overline{N}_i\,, \qquad i = 1,2\,. \tag{10}$$

The limiting amount of extracted capital for the system E_∞ can be found from the conditions (58), (61). The profitability of the system is determined by the equality (58).

2.1.2. System without market

In this case when $t \to \infty$ the resource estimates in subsystems turned out to be the same and equal to some value $\overline{p}$. Instead of equation (56) here we have

$$p_i(\overline{N}_i, \overline{M}_i) \;=\; \overline{p}\,, \qquad i = 1, \ldots, k\,. \tag{11}$$

The value of $\overline{p}$ is to be found from the condition that the intermediary's resource stock does not change (that is, it sells everything it buys)

$$\sum_{i=1}^{k} (\overline{N}_i - N_{i0}) \;=\; 0\,. \tag{12}$$

The equalities (62), (63) jointly with the equations (57) allow us to find vectors $\overline{N}$, $\overline{M}$ and $\overline{p}$, which determine the maximum of the extracted capital E_∞ in this case.

EXAMPLE 2 Consider the same system as in the previous Example. The initial stocks of resorce and capital N_{10}, N_{20}, M_{10}, M_{20} are given. The dependencies of the estimates on the current stocks M_i and N_i $(i = 1, 2)$ have the form

$$p_1 = \alpha \frac{M_1}{N_1}, \qquad p_2 = \beta \frac{M_2}{N_2}.$$

For these dependencies the equations (57) take the form

$$\begin{aligned} \frac{dM_1}{dN_1} &= -\alpha \frac{M_1}{N_1}, \qquad & M_1(N_{10}) &= M_{10}, \\ \frac{dM_2}{dN_2} &= -\beta \frac{M_2}{N_2}, \qquad & M_2(N_{20}) &= M_{20}. \end{aligned}$$

Solution of these equations gives $M_1(N_1)$ and $M_2(N_2)$

$$M_1 = \frac{M_{10} \cdot N_{10}^{\alpha}}{N_1^{\alpha}}, \qquad M_2 = \frac{M_{20} \cdot N_{20}^{\beta}}{N_2^{\beta}}.$$

The conditions (62) and (63) allow us to find the resource stocks for each of the agents $\overline{N}_i$ after exchange is completed. These conditions can be rewritten as

$$\begin{cases} N_{10} + N_{20} = \overline{N}_1 + \overline{N}_2, \\ \alpha \dfrac{M_{10} N_{10}^{\alpha}}{\overline{N}_1^{\alpha+1}} = \beta \dfrac{M_{20} N_{20}^{\beta}}{\overline{N}_2^{\beta+1}}. \end{cases}$$

For the particular case of $\alpha = \beta = \gamma$, this system has the following solution

$$\begin{aligned} \overline{N}_1 &= \frac{(N_{20} + N_{10}) \cdot (M_{10} N_{10}^{\gamma})^{\frac{1}{1+\gamma}}}{(M_{10} N_{10}^{\gamma})^{\frac{1}{1+\gamma}} + (M_{20} N_{20}^{\gamma})^{\frac{1}{1+\gamma}}}, \\ \overline{N}_2 &= \frac{(N_{20} + N_{10}) \cdot (M_{20} N_{20}^{\gamma})^{\frac{1}{1+\gamma}}}{(M_{10} N_{10}^{\gamma})^{\frac{1}{1+\gamma}} + (M_{20} N_{20}^{\gamma})^{\frac{1}{1+\gamma}}}. \end{aligned}$$

The values of $\overline{N}_1$ and $\overline{N}_2$ determine the equilibrium stocks of capital

$$\begin{aligned} \overline{M}_1 &= (M_{10} \cdot N_{10}^{\gamma})^{\frac{1}{1+\gamma}} \cdot W, \\ \overline{M}_2 &= (M_{20} \cdot N_{20}^{\gamma})^{\frac{1}{1+\gamma}} \cdot W, \end{aligned}$$

where

$$W = \left(\frac{(M_{10} N_{10}^{\gamma})^{\frac{1}{1+\gamma}} + (M_{20} N_{20}^{\gamma})^{\frac{1}{1+\gamma}}}{N_{20} + N_{10}} \right)^{\gamma}.$$

The substitution of these expressions into (58) yields the profitability of the system.

If resource exchange takes place without intermediary then the total resource stock and capital stock in the system do not change and the single resource estimate $\overline{p}$ is established in the system. The equations (57) take the form

$$\frac{dM_i}{dN_i} = -\overline{p}\,, \qquad i = 1, \ldots, k\,,$$

So

$$\overline{M_I} = M_{i0} - \overline{p}(\overline{N} - N_{i0})\,, \qquad i = 1, \ldots, k\,.$$

The substitution of these expressions into the conditions (62), (63) yields $\overline{N_i}$ and $\overline{p}$.

2.2. The duration of resource exchange is limited

We assume now that the duration of resource exchange is given and equal to τ. In this case the intermediary has to increase the prices which it offers to sellers and to decrease the prices it offers to buyers, compared to the equilibrium estimates p_i. This leads to the irreversible losses and reduces the amount of capital extracted from the system. The maximal possible value of this amount of capital E_τ here turns out to be lower than E_∞. The difference between these two values

$$\Delta S = (E_\infty - E_\tau) > 0 \qquad (13)$$

characterizes the irreversibility of the resource exchange process.

2.2.1. The condition of optimal trading

Consider exchange between intermediary and EA. The problem is how to control the offered price of resource in such a way that in a given time interval τ the resource stock ΔN is sold with the maximal profit. It is clear that the same conditions, which determine solution to this problem, also determine the solution of the problem when intermediary buys resource from EA and tries spend the minimal amount of capital for that. In both cases the EA's amount of capital at the end of the process $M(\tau)$ must be minimal.

Formally, the problem is written as

$$\overline{M} = M(\tau) \quad \to \quad \min_c \qquad (14)$$

subject to constraints

$$\overline{N} = N(\tau) = N_0 - \Delta N\,, \qquad (15)$$

$$\frac{dM}{dN} = -c\,, \qquad (16)$$

$$\int_0^\tau dt = \int_{\overline{N}}^{N_0} \frac{dN}{n(p(N,M),c)} = \tau\,. \qquad (17)$$

In order to replace the problem independent variable dt with dN we use the dependence

$$\frac{dN}{dt} = -\, n(p,c)\,,$$

in which n is a nonzero function over the interval $(0,\tau)$.

In the problem (65), (67), (68) one is required to find such dependence $c^*(N)$, for which the capital increment for EA is minimal.

The conditions of optimality for this problem were derived in [12]. They have the following form

$$\frac{d}{dN}\left[\frac{\partial n/\partial c}{n^2(p,c)}\right] = \frac{\partial n/\partial p\cdot(\partial p/\partial M)}{n^2(p,c)}\,. \tag{18}$$

If resource estimate p depends only on its stock N and $\partial p/\partial M = 0$, then the condition (69) becomes simpler

$$\frac{\partial n/\partial c}{n^2(p,c)} = \text{const}\,. \tag{19}$$

If

$$n(p,c) = \alpha(c-p)\,, \tag{20}$$

then from the condition (70) during the sale of the resource we get

$$c^*_\tau(N,\overline{N}) = p(N) - \frac{\overline{N}-N_0}{\alpha\tau}\,, \tag{21}$$

and the capital received from the sale is

$$E_\tau(\overline{N}) = E_\infty(\overline{N}) - \frac{(\overline{N}-N_0)^2}{\alpha\tau}\,, \tag{22}$$

where E_∞ is the capital the intermediary could receive if $\tau\to\infty$, and it sells it using the equilibrium prices $c(N)=p(N)$. The function $E(\tau)$ is shown in Fig. 1.

If $\tau<\tau_0=\varDelta N^2/(\alpha E_\infty)$ then the intermediary has to subside the customer. If $\tau^*=2\tau_0$ then the average rate of profit $e(\tau)=E(\tau)/\tau$ is maximal and equal to

$$e^* = \frac{\alpha}{4}\Big[\frac{E_\infty(\overline{N}-N_0)}{\overline{N}-N_0}\Big]^2.$$

The loss of capital in comparison with the equilibrium process gives an estimate of the reduction of profitability (capital dissipation).

$$\sigma = n(p,c)(p-c)\,. \tag{23}$$

The amount of dissipative losses is determined as

$$\varDelta S(\tau) = \int_0^\tau \sigma(t)\,dt = \int_0^\tau n(p,c)(p-c)\,dt\,. \tag{24}$$

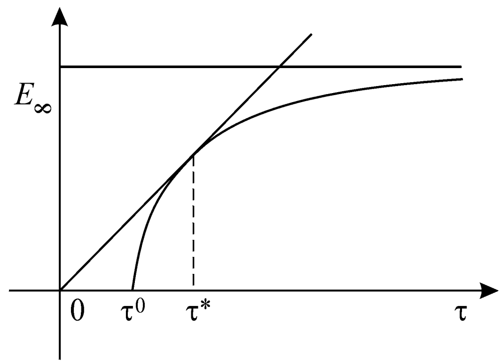

Fig. 1: The dependence of the profitability on the duration of the process.

For the above-considered example this loss is

$$\Delta S(\tau,\overline{N}) \;=\; \int_0^\tau \alpha(p(N)-c(N))^2 dt \;=\; \frac{(\overline{N}-N_0)^2}{\alpha\tau}\,,$$

so

$$E(\tau) \;=\; E_\infty(\overline{N}) - \Delta S(\tau,\overline{N}) \;=\; E_\infty(\overline{N}) - \int_0^\tau n(p,c)(p-c)\,dt\,. \tag{25}$$

The expression (76) holds for arbitrary dependence $n(p,c)$. Indeed, when dt is replaced with dN, the integral in (75) can be rewritten as

$$\Delta S(\overline{N}) \;=\; S(\overline{N}) - S(N_0) \;=\; \int_{N_0}^{\overline{N}} (p(N) - c_\tau(N,\overline{N}))\,dN\,.$$

In its turn, the extracted capital is

$$E(\tau,\overline{N}) \;=\; \int_{N_0}^{\overline{N}} c_\tau(N,\overline{N})\,dN\,, \qquad E_\infty(\overline{N}) \;=\; \int_{N_0}^{\overline{N}} p(N)\,dN\,. \tag{26}$$

The comparison of these two expressions yields (76). Thus, the optimal process of buying (selling) corresponds to the process of minimal dissipation of capital.

2.2.2. Extraction of the maximal profit

In this case the problem is reduced to trading (buying or selling) resource from each of the EA. The trading process must proceed optimally from the viewpoint of capital extraction (spending), so that the price c and the resource estimate p must obey the conditions (69), (70) at any moment of time. The volumes ΔN_i bought from each of the m subsystems must be chosen optimally and obey the condition

$$\sum_{i=1}^{m} \overline{N}_i \;=\; \sum_{i=1}^{m} N_{i0}\,. \tag{27}$$

Market can be viewed as one of the subsystems, for which the estimate p_- does not depend on neither the resource stock nor the capital stock. Therefore, for all dependencies $n(c, p_-)$ the optimal price c for buying and selling on such a market must be time-independent.

Thus, the problem of extracting the maximal possible amount of capital in a closed microeconomic system in finite time is reduced to the two staged process. During the first stage the solutions of m problems (65)–(68) about the optimal trading (buying and selling) for each of the subsystems for fixed initial and final stocks of (N_{i0} and $\overline{N_i}$) are found. The maximal amount of capital extracted (or minimal amount of capital spent) $E_i(\tau)$ depends on $\overline{N_i}$. On the second stage it is necessary to find the optimal $\overline{N_i}$ from the condition

$$\sum_{i=1}^{m} E_i(\tau, \overline{N_i}) \quad \to \quad \max_{\overline{N}_i} \tag{28}$$

subject to condition (78), which leads to the equality

$$\frac{\partial E_i(\tau, \overline{N_i})}{\partial \overline{N_i}} \;=\; \varLambda\,, \qquad i = 1, \ldots, m\,,$$

where $\varLambda$ is found from (78).

After taking into account (77) we get

$$\frac{\partial E_i(\tau, \overline{N}_i)}{\partial \overline{N}_i} \;=\; c_{i\tau}(\overline{N}_i, \overline{N}_i) + \int\limits_{N_{i0}}^{\overline{N}_i} \frac{\partial c_{i\tau}(N_i, \overline{N}_i)}{\partial \overline{N}_i}\, dN_i \;=\; \overline{c}_{i\tau}(\overline{N_i})\,. \tag{29}$$

The first summand in the first part is actually the optimal price at time τ, and the second summand is a correction on this price. It is determined by the averaged sensitivity of the optimal price to the volume of sold (or bought) resource. We shall call the expression (80) the corrected price. The condition of optimal choice of the trade volume takes the form of equality of the corrected prices for all subsystems

$$\overline{c}_{i\tau}(\overline{N_i}) \;=\; \varLambda\,, \qquad i = 1, \ldots, m\,. \tag{30}$$

EXAMPLE 3 Assume that for each subsystem

$$p_i \;=\; \frac{h_i}{N_i}\,, \qquad\qquad i = 1, \ldots, m\,, \tag{31}$$

$$n_i(c, p) \;=\; \alpha_i(c_i - p_i)\,, \qquad i = 1, \ldots, m\,. \tag{32}$$

Let us find out what amount of capital can be extracted from the i-th subsystem in infinite time. After taking into account (82), from (77) it follows

$$E_{i\infty}(\overline{N}_i) \;=\; h_i \int_{N_i 0}^{\overline{N}_i} \frac{dN_i}{N_i} \;=\; h_i \ln \frac{\overline{N}_i}{N_{i0}}\,, \qquad i = 1, \dots, m\,.$$

According to equality (73)

$$E_i(\tau, \overline{N}_i) \;=\; h_i \ln \frac{\overline{N}_i}{N_{i0}} - \frac{(\overline{N}_i - N_{i0})^2}{\alpha_i \tau}\,. \tag{33}$$

The condition (81), which determines the optimal choice of $\overline{N_i}$, takes the form (see also (72))

$$\overline{c}_{i\tau}(\overline{N}_i) \;=\; \left[p_i(\overline{N}_i) - \frac{\overline{N}_i - N_{i0}}{\alpha\tau}\right] - \frac{\overline{N}_i - N_{i0}}{\alpha\tau} \;=\; \Lambda\,. \tag{34}$$

This problem becomes much simpler if all subsystems have constant estimates $p = \text{const}$. Then the condition of optimality (85) leads to the equalities

$$p_i - \frac{2}{\alpha_i\tau}(\overline{N}_i - N_{i0}) \;=\; \Lambda \quad \rightarrow \quad \Delta N_i \;=\; \frac{\alpha_i \tau}{2}(p_i - \Lambda)\,. \tag{35}$$

From the condition (78) it follows that the value of Λ is equal to the averaged weighted resource estimate

$$\Lambda \;=\; \frac{\sum\limits_{i=1}^{m} \alpha_i p_i}{\sum\limits_{i=1}^{m} \alpha_i}\,,$$

and

$$\overline{N}_i^* \;=\; \frac{\tau\alpha_i}{2}\left(p_i - \frac{\sum\limits_{\nu=1}^{m} \alpha_\nu p_\nu}{\sum\limits_{\nu=1}^{m} \alpha_\nu}\right) + N_{i0}\,. \tag{36}$$

After the substitution of $\overline{N}_i^*$ into (86) we get the expression for the maximal possible amount of capital $E_i(\tau, \overline{N}_i^*)$, which can be extracted from the subsystem in time τ. The profitability of the system is

$$E_\tau^* \;=\; \sum_{i=1}^{m}\left[p_i(\overline{N}_i^* - N_{i0}) - \frac{(\overline{N}_i^* - N_{i0})^2}{\alpha_i\tau}\right]. \tag{37}$$

3. Open Microeconomic System

3.1. The conditions of equilibrium of an open microeconomic system without intermediary

Consider a system which includes r markets with perfect competition and $(k - r)$ economic agents which exchange resources and capital with each other (Fig. 2).

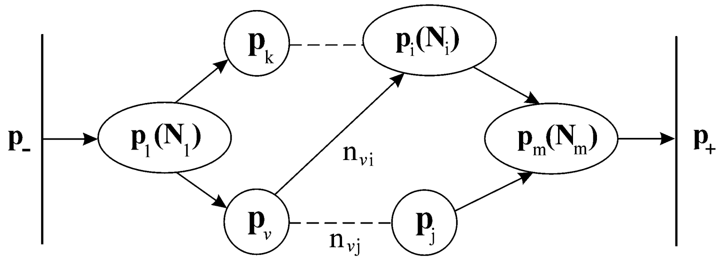

Fig. 2: An open microeconomic system.

The resource estimates for the markets p_i $(i = 1, \ldots, r)$ are constant. For $i > r$ these estimates depend on the stocks of resource N_i and capital M_i in the i-th subsystem. The initial values of these variables for $i > r$ are given. We assume that the capital estimates are constant and are the same for each subsystem and find the equilibrium stocks in subsystems and fluxes between them. Let the flux of resource n_i between economic agent with resource estimate $p_i(N_i, M_i)$ be determined by the flux of capital $q_i = -n_i c_i$, where c_i is the price of resource. This price is higher than p_i, if the economic agent sells resource $(n_i < 0)$, and it is lower than p_i, if $n_i > 0$. If the exchange takes place between economic agent and the j-th market then $c_i = p_j$, and

$$n_{ij} = n_{ij}(p_j, p_i)\,, \quad q_{ij} = -p_j n_{ij}\,, \quad j = 1, \ldots, r\,, \quad i = r+1, \ldots, k\,. \tag{38}$$

If exchange takes place between two economic agents, then let us define (following [4]) the price $c_{i\nu}$ in such a way that

$$\widetilde{n}_{i\nu}(p_i, c_{i\nu}) \;=\; -\widetilde{n}_{\nu i}(p_\nu, c_{i\nu})\,, \tag{39}$$

where $(i;\nu) \geq r+1$. The conditions (90) allow us to express $c_{i\nu}$ in terms of p_i, p_ν. As a result we get

$$n_{i\nu}(p_i, p_\nu) \;=\; -n_{\nu i}(p_\nu, p_i)\,. \tag{40}$$

For example, let

$$\widetilde{n}_{i\nu} \;=\; \widetilde{\alpha}_{i\nu}(p_i - c_{i\nu})\,, \qquad \widetilde{n}_{\nu i} \;=\; \widetilde{\alpha}_{\nu i}(p_\nu - c_{i\nu})\,.$$

From the condition (90) we get $c_{i\nu}$ and the fluxes, that are used in equality (91),

$$c_{i\nu} \;=\; \frac{\tilde{\alpha}_{i\nu} p_i + \tilde{\alpha}_{\nu i} p_\nu}{\tilde{\alpha}_{i\nu} + \tilde{\alpha}_{\nu i}}\,, \tag{41}$$

$$\left.\begin{aligned} n_{i\nu}(p_i, p_\nu) \;&=\; \frac{\tilde{\alpha}_{\nu i}\tilde{\alpha}_{i\nu}}{\tilde{\alpha}_{i\nu} + \tilde{\alpha}_{\nu i}}(p_i - p_\nu) \;=\; \alpha_{i\nu}(p_i - p_\nu)\,, \\ n_{\nu i}(p_\nu, p_i) \;&=\; -n_{i\nu}(p_i, p_\nu) \;=\; \alpha_{i\nu}(p_\nu - p_i)\,. \end{aligned}\right\} \tag{42}$$

The fluxes of capital are

$$q_{i\nu}(p_i, p_\nu) \;=\; -c_{i\nu}(p_i, p_\nu) n_{i\nu}(p_i, p_\nu) \;=\; -q_{\nu i}(p_\nu, p_i)\,. \tag{43}$$

The dynamics of this system is governed by the following differential equations

$$\frac{dN_i}{dt} = \sum_{j=1}^{k} n_{ji}(p_j, p_i)\,, \quad N_i(0) = N_{i0}\,, \tag{44}$$

$$\frac{dM_i}{dt} = \sum_{j=1}^{k} q_{ji}(p_j, p_i)\,, \quad M_i(0) = M_{i0}\,, \quad i = r+1, \ldots, m\,, \tag{45}$$

here $n_{ii} = q_{ii} = 0$. In equilibrium we have $(k - r)$ conditions

$$\sum_{j=1}^{k} n_{ji}(p_j, p_i) = 0\,, \qquad i = r+1, \ldots, k\,, \tag{46}$$

which determine the equilibrium resource estimates p_i^0 for $i = r+1, \ldots, k$.

Because the resource and capital fluxes are related to each other via (94), where $c_{i\nu} > 0$, the constancy of the resource vector N and the constancy of capital estimates lead to the constancy of the vector of capital M. From the stability of the state of equilibrium of the system (95) follows the stability of the equilibrium in the system (96).

Note that when c is chosen from the condition (90) the capital dissipation is

$$\sigma_{i\nu} = n_{i\nu}(p_i, c_{i\nu})(p_i - c_{i\nu}) + n_{\nu i}(p_\nu, c_{i\nu})(c_{i\nu} - p_\nu) = n_{i\nu}(p_i, p_\nu)(p_i - p_\nu)\,. \tag{47}$$

The conditions of equilibrium for the system (97) determinethe values of the estimates p_i $(i = 1, \ldots, k)$ only. Since these estimates depend on N_i and M_i, a subset exists in the state space with coordinates N and M, which corresponds to the equilibrium of open microeconomic system.

If the system is close to equilibrium then the continuous and smooth dependencies $n_{i\nu}$ can be approximated as (93). In this case the dissipation σ can be rewritten as

$$\sigma = \frac{1}{2}\sum_{j=1}^{k}\sum_{i=1}^{k} \alpha_{ij}(p_i - p_j)^2\,,$$

and the conditions of equilibrium as

$$\sum_{j=1}^{k} \alpha_{ji}(p_i - p_j) = 0\,, \qquad i = r+1, \ldots, k\,.$$

σ is strictly convex function and conditions of its minimum on p_i $(i = r+1, \ldots, k)$ are identical with conditions of equilibrium.

Thus: *if the laws of resource exchange are close to linear in an open microeconomic system then in equilibrium the resource and capital stocks are distributed between economic agents in such a way that capital dissipation σ is minimal.*

This statement is an analogue of the Prigogine theorem in irreversible thermodynamics.

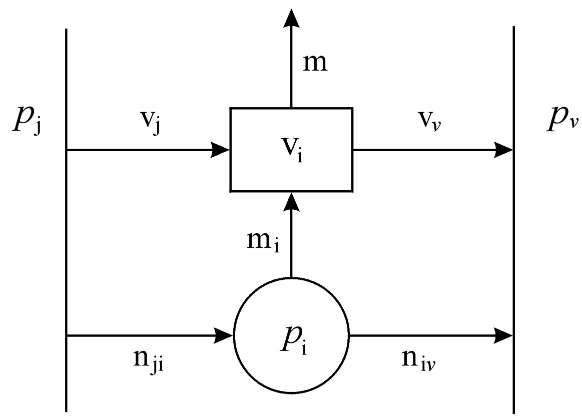

Fig. 3: An open microeconomic system with an intermediary.

3.2. Limiting rate of capital extraction

Let the system considered above include a intermediary which can buy a resource from some economic agents and sell it to other ones. By operating in this way it will extract capital (Fig. 3). The intermediary sets the price v_i when it makes an exchange with the i-th subsystem, and the flux of resource here is $m_i(p_i, v_i)$. In equilibrium the problem of maximal rate of capital extraction takes the form

$$m = -\sum_{i=1}^{k} m_i(p_i, v_i) v_i \quad \to \quad \max_{v,p} \tag{48}$$

subject to constraints

$$\sum_{i=1}^{k} m_i(p_i, v_i) = 0\,, \tag{49}$$

$$\sum_{j=1}^{k} n_{ji}(p_j, p_i) = m_i(p_i, v_i)\,, \quad i = r+1, \ldots, k\,. \tag{50}$$

The minus sign appears in (99) because we assign as positive the resource exchange flux, which goes from EA to intermediary. Such flux is accompanied by spending of capital. The condition (100) corresponds to the resource balance for the intermediary, and the conditions (101) correspond to the balances for each of the $(k-r)$ EA.

In order to obtain the conditions of optimality for the problem (99)–(101) we write down its Lagrange function

$$L = \sum_{i=1}^{k} \left[m_i(p_i, v_i)(\Lambda - v_i + \lambda_i) - \lambda_i \sum_{j=1}^{k} n_{ji}(p_j, p_i) \right]. \tag{51}$$

Here $\lambda_i = 0$ for $i \le r$.

The conditions of optimality have the form

$$\frac{\partial L}{\partial v_i} = 0 \quad \Longrightarrow \quad \frac{\partial m_i}{\partial v_i}(\Lambda - v_i - \lambda_i) = m_i(p_i, v_i)\,, \quad i = 1, \ldots, k\,, \tag{52}$$

$$\frac{\partial L}{\partial p_i} = 0 \quad \Longrightarrow \quad \frac{\partial m_i}{\partial p_i}(\Lambda - v_i - \lambda_i) = \lambda_i \sum_{j=1}^{k} \frac{\partial n_{ji}}{\partial p_i}\,, \quad i = r+1, \ldots, k\,. \tag{53}$$

The conditions (100), (101), (103), (104) determine $2(k-r)$ unknowns p_i and λ_i, the value of Λ and k optimal prices v_i.

In particular, if $n_{ji} = \alpha_{ji}(p_i - p_j)$, $m_i = \alpha_i(v_i - p_i)$, then these conditions can be rewritten as

$$\sum_{i=1}^{m} \alpha_i(v_i - p_i) \;=\; 0\,, \tag{54}$$

$$\sum_{j=1}^{k} \alpha_{ji}(p_i - p_j) \;=\; \alpha_i(v_i - p_i)\,, \quad i = r+1, \ldots, k\,, \tag{55}$$

$$2v_i \;=\; \lambda_i + \Lambda + p_i\,, \quad i = 1, \ldots, k\,, \tag{56}$$

$$-\alpha_i(\Lambda - v_i + \lambda_i) \;=\; \lambda_i \sum_{j=1}^{k} \alpha_{ji}\,, \quad i = r+2, \ldots, k\,. \tag{57}$$

EXAMPLE 4 Let us consider the particular case when $k = r = 2$, $p_1 = p_-$, $p_2 = p_+$, and $p_+ > p_-$. Then the system (105)–(107) takes the following form (the conditions (106) and (108) are omited since $r = k$ and λ_1 and λ_2 are equal zero)

$$\begin{aligned} \alpha_1(v_1 - p_-) + \alpha_2(v_2 - p_+) &= 0\,, \\ 2v_1 &= \Lambda + p_-\,, \\ 2v_2 &= \Lambda + p_+\,. \end{aligned}$$

The unknowns here are v_1 and v_2.

This problem has the following solution

$$\begin{aligned} v_1^* &= \frac{2\alpha_1 p_- + \alpha_2(p_+ + p_-)}{2(\alpha_1 + \alpha_2)}\,, \\ v_2^* &= \frac{2\alpha_2 p_+ + \alpha_1(p_+ + p_-)}{2(\alpha_1 + \alpha_2)}\,. \end{aligned}$$

Here v_1^* is the optimal price of buying the resource, v_2^* is the optimal price of selling it. After taking into account the optimality of prices from condition (99) it follows that the maximal rate of extraction of the basic resource is

$$m^* \;=\; \frac{\alpha_1\alpha_2(p_+ - p_-)^2}{4(\alpha_1 + \alpha - 2)}\,.$$

Consider another particular case when $r = 2$, and $k = 3$ (Fig. 3). In other words, the system contains two markets, an intermediary and an economic agent. Here v_3 is the price of buying (selling) resource by intermediary from the EA and p_3 is the estimate of resource by EA. The subsystem contacts with the markets and the intermediary and the capital and resource fluxes occur between them. The goal of the intermediary remains the same — extraction of the maximal-possible amount of capital. In order to solve this problem we rewrite the system (105)–(108) in the following form

$$\alpha_1(v_1 - p_-) + \alpha_2(v_2 - p_+) + \alpha_3(v_3 - p_3) \ = \ 0\,, \tag{58}$$

$$\alpha_3(v_3 - p_3) \ = \ \alpha_4(p_3 - p_-) + \alpha_5(p_3 - p_+)\,, \tag{59}$$

$$v_1 \ = \ \frac{\Lambda + p_-}{2}\,, \tag{60}$$

$$v_2 \ = \ \frac{\Lambda + p_+}{2}\,, \tag{61}$$

$$v_3 \ = \ \frac{\lambda_3 + \Lambda + p_3}{2}\,, \tag{62}$$

$$-\alpha_3(\Lambda - v_3 + \lambda_3) \ = \ \lambda_3(\alpha_4 + \alpha_5)\,. \tag{63}$$

This gives v_1^*, v_2^*, v_3^*, p_3^*.

Let us investigate the dependence of the limiting rate of profit and the flux of resource between the intermediary and EA on the coefficient α_4. We set

$$\alpha_1 = 0.2\,, \quad \alpha_2 = 0.3\,, \quad \alpha_3 = 0.01\,, \quad \alpha_5 = 0.4\,, \quad p_- = 4\,, \quad p_+ = 7$$

and will find for them the dependencies $v_1^*(\alpha_4)$, $v_2^*(\alpha_4)$, $v_3^*(\alpha_4)$ and $p_3^*(\alpha_4)$. The substitution of the values found into equations for the limiting rate of profit (99) yields the function $m^*(\alpha_4)$.

The function $m^*(\alpha_4)$ and the dependence of the flux between the intermediary and EA $m_3^*(\alpha_4) = \alpha_3 \cdot (v_3^*(\alpha_4) - p_3^*(\alpha_4)$ are shown in Fig. 4.

From these figures it is clear that the function $m^*(\alpha_4)$ attains minimum for some $\alpha = \alpha_m$, and the function $m_3^*(\alpha_4)$ is equal to zero for the same value α_4 only.

This is natural because when the intermediary conducts exchanges with EA its objective is to enhance this profit. If the exchange flux is to equal zero, then $m_3 = 0$ for some value of α_4 and the objective flux m is minimal.

4. Conclusion

As a conclusion to the analysis of the maximal work problems in thermodynamics and maximal profit problems in microeconomics, we will now try to trace the link between the second law of thermodynamics and its analogues in microeconomics. Among numerous formulations of the second law of thermodynamics we will first consider the Clausius formulation, which was made more accurate by Planck: *"It*

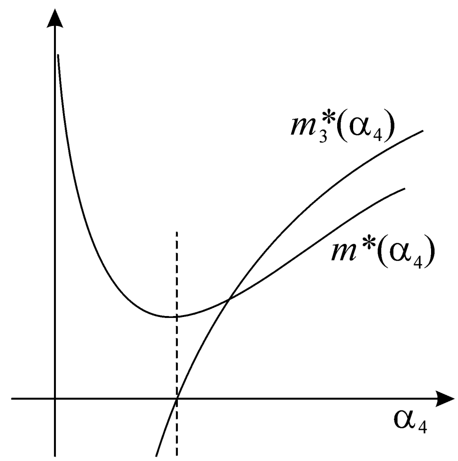

Fig. 4: The dependencies of the limiting rate of profit and the flux of capital between intermediary and EA on the parameter α_4.

is impossible to transfer heat from a cooler body to a hotter body and produce no other effects", and the Leontovich formulation that: *"It is not possible to build a device which would produce positive work only by cooling one body without any other effects"*.

In microeconomics these formulations correspond to the following statements:

1. *The flux of resource cannot flow from an economic agent with the higher estimate to an economic agent with the lower estimate, without other changes taking place.*
2. *It is not possible to produce profit by exchange with one economic agent without any other changes.*

The quantitative characteristics of the second law of thermodynamics for heat systems is the equality to zero in the cyclic reversible process of the Clausius integral

$$\oint \frac{dQ}{T} \;=\; 0\,. \tag{64}$$

From this equality it follows that such a function S exist, whose differential is $dS = dQ/T$, and whose increment depends only on the initial and final states of the homogeneous system and does not depend on the path between the initial and the final states

$$S \;=\; S_0 + \int\limits_{Q_0}^{Q} \frac{dQ}{T}\,. \tag{65}$$

In irreversible process the entropy of the system can only increase and its exergy can only decrease. The state of equilibrium of a closed thermodynamic system corresponds to the maximum of its entropy and minimum of its exergy, subject to the system constraints.

In order to obtain an analogue of entropy we consider a circular process in uniform microeconomic system. The state of this system is described by the the stock of resource N and the stock of capital M, p^0 is an equilibrium resource estimate in units of M and r is the estimate of capital. In a circular reversible exchange process between EA and intermediary, the intermediary can not produce profit because its states and the states of the EA are the same in the beginning and in the end of the process

$$\oint dS \;=\; \oint r(dM + p^0 dN) \;=\; \oint (dE + p^0 dN) \;=\; 0\,.$$

Thus a function $S(N, M)$ exists, whose increment in a reversible process depends only on the initial and the final states and does not depend on the trajectory between these states.

In irreversible process the purchase of resource ($dN > 0$) takes place if $p \geq p^0$, and its sail ($dN < 0$) if $p \leq p^0$; the value of the integral obeys

$$\int_{N_0}^{N} p\, dN \;\geq\; \int_{N_0}^{N} p^0 dN\,,$$

and the difference

$$\Delta S \;=\; \Delta E^0 - \Delta E \;=\; \int_{N_0}^{N} (p - p^0)\, dN \;\geq\; 0\,.$$

This difference characterizes the irreversible losses of capital and is an analogue of the entropy increment in thermodynamics. The resource exchange processes in a closed microeconomic system are accompanied by equalization of the resource estimates of economic agents. Therefore, they are accompanied by reduction in profitability.

As a consequence of the second law of thermodynamics Max Planck formulated the following statement: “Each natural process proceeds in such direction that the sum of entropies of all the bodies which participate in it increases”. Exactly the same statement is true in the irreversible microeconomics: “Each resource exchange process proceeds in such direction that the net loss of profit of participating economic agents is positive. In equilibrium the profitability of a closed economic system attains maximum allowable by the constraints imposed on the system”. The capital dissipation $\sigma \geq 0$, which has been introduced in this paper, describes the rate of irreversible losses of system profitability.

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